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Numerical methods for solving singular integral equations with Cauchy-type kernels

Dr. Mathibele Nchabeleng, UP

Overview

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2. Motivation
3. Study Problem

Introduction and background

Introduction and background

Integral Equations

- Integral equations are equations in which some unknown function to be determined appears under one or several integral signs [5, 6].
- The name integral equation was given by du Bois-Reymond in 1888.
- There are many types of integral equations.
- The classification of integral equations depends mainly on the limits of integration and the kernel of the equation.

Introduction and background

Integral Equations

- Integral equations arise in several fields of science; for example, in elasticity, potential theory, fluid mechanics, biomechanics, approximation theory, plasticity, game theory, queuing theory, medicine, acoustics, heat and mass transfer, economics [4].
- More details about integral equations and their origins can be found in [2, 3].

Introduction and background

Integral Equations

- We will focus our concerned on equations of the form:

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x} \mathrm{d}t + \int_{-1}^{1} k(x,t)\varphi(t)\mathrm{d}t = f(x), \quad -1 < x < 1, \tag{1}$$

  where the kernel function $k(x,t)$ and the forcing function $f(x)$ are prescribed and the function $\varphi(t)$ is the unknown function to be determined.

- Equation (1) is called Cauchy-type singular integral equation of the first kind and presents a Cauchy-type singularity at $t = x$.

- Singular integral equations with Cauchy kernels appear in many practical problems of elasticity, crack theory, wing theory and fluid flow [1].

Introduction and background

Integral Equations

- The simplest singular integral equation of the first kind has the form

$$\fint_{-1}^{1} \frac{\varphi(t)}{t-x}\mathrm{d}t = f(x), \quad -1 < x < 1. \tag{2}$$

- Equation (2) is called the characteristic singular integral equation and it is obtained when $k(x,t) = 0$ in (1).

- The integral in (2) is understood in the principal value sense and is defined as

$$\fint_{-1}^{1} \frac{\varphi(t)}{t-x}\mathrm{d}t = \lim_{\varepsilon\to 0}\left[\int_{-1}^{x-\varepsilon} \frac{\varphi(t)}{t-x}\mathrm{d}t + \int_{x+\varepsilon}^{1} \frac{\varphi(t)}{t-x}\mathrm{d}t\right], \quad -1 < x < 1. \tag{3}$$

Introduction and background

Integral Equations

- The closed-form solution of the characteristic singular integral equation (2), which is unbounded at both end-points $x = \pm 1$, is given by the formula

$$\varphi(x) = -\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}f(t)}{t-x}\mathrm{d}t + \frac{C}{\pi\sqrt{1-x^2}}, \tag{4}$$

where

$$C = \int_{-1}^{1}\varphi(t)\mathrm{d}t. \tag{5}$$

- Equation (2) can also be solved to obtain an approximate analytical solution and a numerical solution.

Motivation

Motivation

A hydraulic fracture problem

- In recent years, we have started to model the elasticity equation of hydraulic fracture problems using the KGD elasticity model which resulted in the problem of solving a system of ordinary integro-differential equations given by

$$-\frac{1}{3}f^3\frac{\mathrm{d}^2P}{\mathrm{d}\xi^2} - f^2\frac{\mathrm{d}f}{\mathrm{d}\xi}\frac{\mathrm{d}P}{\mathrm{d}\xi} - c_1\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} + c_2 f(\xi) + c_3 g(\xi) = 0, \qquad (6)$$

$$P(\xi) = -\frac{2}{\pi}\int_0^1 \frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\eta}{\eta^2-\xi^2}\mathrm{d}\eta, \qquad (7)$$

$$-P(\xi) \to \frac{\gamma}{[2(\xi-1)]^{\frac{1}{2}}} \quad \text{as} \quad \xi \to 1^+, \qquad (8)$$

$$f(0) = 1, \quad f(1) = 0, \quad f'(0) = 0. \qquad \text{(9a-c)}$$

Study Problem

Study Problem

Cauchy-type singular integral equation

- Consider the problem of solving the singular integral equation given by

$$\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 \frac{\varphi_\xi(\xi)}{\xi - x}\mathrm{d}\xi\right) = 1, \qquad 0 \leqslant x \leqslant 1, \tag{10}$$

subject to the boundary conditions

$$\varphi_x(0) = 0, \quad \varphi(1) = 0. \tag{11a-b}$$

- We want to solve (10) subject (11a-b) to both analytically and numerically.

QUESTIONS?

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